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HAMILTON'S PRINCIPLE AND CERTAIN PROBLEMS OF DYNAMICS OF PERFECT FLUID*

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From the equation of motion of perfect incompressible homogeneous fluid are derived the Hamilton variational equation and the Lagrange function for some vortex flows of fluid in a multiply connected region, of arbitrary potential flow in boundless region in the presence of singularities inside the stream, of flow over a wing with circulation in unstable stream whose velocity is an arbitrary meromorphic function. The Lagrange function is defined in explicit form as a functional of the boundary of the fluid and its normal velocity, which enables us to solve some problems of dynamics of certain bodies in fluid, of vortex flows and flows in the presence of a free boundary by methods on analytic mechanics.

1. From the history of the problem. Not withstanding the efforts of many mathematicians, the problem of obtaining the equations of motion in a potential flow of a perfect incompressible fluid remained unsolved up to 1867. Suddenly Thomson and Tait /1/ found a simple solution of this problem, applying to the fluid the Hamilton principle

$$\int_{t_1}^{t_2} dt \int_{\partial\Omega} p\delta n\, dS = \int_{t_1}^{t_2} dt\delta L \qquad (1.1)$$
$$\delta n\,(t_1) = \delta n\,(t_2) = 0$$

where δn is the virtual displacement of the boundary of solid body $\partial\Omega$, compatible with kinematic conditions, and the integral of $\partial\Omega$ represents the work of the force of pressure p of fluid on the displacement δn.

The Lagrange function L, as well as the motions of the solid body, is equal to the difference of the fluid kinetic energy T and the potential energy of the system

$$L = T - \Pi \qquad (1.2)$$

A system of differential equation follows from (1.1), which is satisfied by N, generalized coordinates of the body (in the general case $N = 6$ for a solid body)

$$-\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} + \frac{\partial L}{\partial q_i} = Q_i, \quad \sum_{i=1}^{N} Q_i\delta q_i = \int_{\partial\Omega} p\delta n\, dS \qquad (1.3)$$

Two problems of dynamics can be solved by using Eqs.(1.3): determination of reactions Q_i acting on the body for a known law of motion $q_i(t)$, and the definition of the law of motion $q_i(t)$ for specified generalized forces Q_i.

In 1869 Kirchhoff provided a convincing proof of this approach /2/. He introduced the Lagrangian displacement of a particle of fluid $\delta\mathbf{x}$ induced by the small perturbation which is determined by the variation $\delta q_i(t)$. At the same time he drew the attention to the fact that for $\delta q_i(t_1) = \delta q_i(t_2) = 0$ it is not possible to assume the Lagrangian displacement of a particle at the instant of time t_2 $(\delta\mathbf{x}(t_2) \neq 0)$. Thus, unlike the points of solid body, the position of particles of fluid is not defined by the instantaneous values of coordinates of a body $q_i(t)$. Hence the Kirchhoff investigation essentially complements the Thomson and Tait idea.

Kirchhoff gave a very simple derivation of the variational equation (1.1) from the Euler equation

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \rho\nabla U, \quad \nabla\mathbf{v} = 0 \qquad (1.4)$$

He transformed the Lagrangian variation of specific energy and, taking advantage of that the $\operatorname{div}\delta\mathbf{x} = 0$, and also of the equation of motion (1.4), obtained

$$\delta\frac{\rho v^2}{2} = \frac{d}{dt}(\rho\mathbf{v}\delta\mathbf{x}) + \operatorname{div}[(p-\rho U)\,\delta\mathbf{x}] \qquad (1.5)$$

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By integrating Eq.(1.5) over the region Ω and over the time, instead of (1.1) we obtain the following equation:

$$\int_{t_1}^{t_2} dt\delta(T-\Pi) = I + \int_{t_1}^{t_2} dt \int_{\partial\Omega} p\delta n\, dS, \quad I = \int_{\Omega} \rho \mathbf{v}\delta\mathbf{x}\, dr \Big|_{t_1}^{t_2} \tag{1.6}$$

By virtue of potential properties of $\mathbf{v} = \nabla\Phi$ integral I reduces to the integral over the surface

$$I = \int_{\Omega} \operatorname{div}(\rho\Phi\delta\mathbf{x})\, d\tau \,|_{t_2} = \int_{\partial\Omega} \rho\Phi\delta x_n\, dS \tag{1.7}$$

where δx_n is the projection of vector $\delta\mathbf{x}$ on the normal n. The boundary $\partial\Omega$ of the body and fluid consists of the same particles of fluid (see the Lagrange theorem in /3/). Hence on it the condition $\delta x_n = \delta n$ is satisfied and by virtue of conditions (1.1) $\delta x_n(t_2) = 0$. From this follows that $I = 0$, and the variational equation (1.1) is obtained. In 1871 Boltzmann drew the attention to that in the multiply connected region it is generally not possible to apply the Gauss theorem in (1.7). It is necessary to introduce mental partitions $\Pi_1, \Pi_2, \ldots, \Pi_{M-1}$, that will transform M-connected region into simply connected one (Fig.1). Then the integral (1.6) is written in the form

$$I = \sum_{k=1}^{M-1} \rho\Gamma_k \int_{\Pi_k} \delta x_n\, dS \tag{1.8}$$

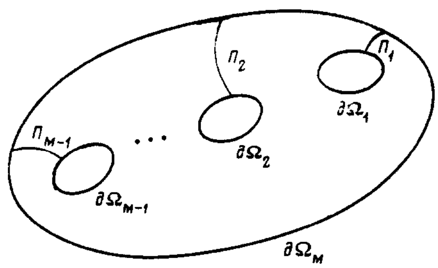

Fig.1

Here Γ_k are the discontinuities of the non-single valued potential Φ on the partition or the circulation velocity on respective contours.

By virtue of $\delta\mathbf{x}(t_2) \neq 0$ the integral (1.8) is generally nonzero, and consequently, the variational equation (1.1) and system (1.3) are not true.

In 1873 Thomson suggested new equations for the potential flow in the multiply connected region /4/. He considered Γ_k as pulses and the respective flow rate γ_k of fluid through the partition Π_k as the conjugate velocities of variation of respective cyclic coordinates. Using the method which was accepted in mechanics as the method of Routh of ignoring cyclic coordinates, Thomson obtained the form of Lagrange function in the variational equation (1.1)

$$L = T - \sum_{k=1}^{M-1} \Gamma_k \gamma_k \tag{1.9}$$

The proof of validity of Lagrange equations (1.3), (1.9) was given by Steklov /5/ and independently by Brian /6/. Recently /7/ a proof was given which is valid for the deformed body in a fluid.

Using the idea of Thomson and Tait, Kirchhoff obtained symmetric equation for the motion of body in a fluid, which are similar to the Euler equations for the motion of solid body in void. The study of the problem of integration of Kirchhoff's equations was commensed in particular cases by Thomson, Tait and Kirchhoff, and were completed by the investigation of Liapunov, Steklov, and Chaplygin.

A considerable interest for application is the work of Zhukovskii /8/, who opened the road for investigation of the problem of motion of a solid body with liquid filling /9 – 12/. Another practically important direction is bound with the work of Sedov /13/ on the dynamic theory of the wing in an unsteady stream with circulation. The basic advantage of his formulas for the force and moment acting on the wing is that the integration contour can be deformed. The calculation of integrals is reduced to the determination of residues near the singular points of integrand functions.

2. Formulations of the Hamilton variational equation and of the Lagrange function. The flow of perfect incompressible homogeneous fluid whose mass forces have a potential U, are considered. The region of flow Ω is bounded from the outside or the inside by the surface $\partial\Omega$. The surface $\partial\Omega$ may consist of several connected parts. The position of surface $\partial\Omega$ or its part is defined by a finite or a denumerable number of parameters $q_1, q_2, \ldots$ (by generalized coordinates). The aim of the present investigation is the derivation of a system of ordinary differential equation (1.3) for the generalized coordinates of the boundary.

Below, the Hamiltonian principle (1.1) is derived from Euler equations (1.4) by the method of Kirchhoff for the following three cases:

1°. The fluid flow region Ω, is a multiply connected bounded by surface $\partial\Omega$ whose position and deformation are defined by the generalized coordinates $q_1, q_2, \ldots$. Three variants are possible: a) the flow is potential, and the potential is multiple-valued function, b) the plane problem, the flow has a constant vortex; c) axisymmetric problem, the flow has a vorticity whose intensity decreases proportionally to the distance from the axis of rotation.

In all three variants the variational equation (1.1) holds and the Lagrange function is

$$L = \int_{\Omega} \rho \left(\frac{v_*^2}{2} - \frac{(\mathbf{v} - \mathbf{v}_*)^2}{2} + U \right) d\tau \tag{2.1}$$

where $\mathbf{v}_*$ is the velocity of field of the imaginary incompressible medium fixed to the boundary $\partial\Omega$. For the cavity in a solid body $\mathbf{v}_*$ is the velocity field of the solid body points. For a potential flow formula (2.1) becomes Thomson formula (1.9). Generally formula (2.1) agrees with the results of /14/.

2°. The body moves in a given potential flow of fluid $\mathbf{v}_0(t, \mathbf{x})$ and creates by that new potential velocity field $\mathbf{v}$ satisfying equations (1.4) and the kinematic conditions at the boundary of the body $\partial\Omega$. The potential velocities $\mathbf{v}$ and $\mathbf{v}_0$ are single valued functions, and at infinity $|\mathbf{v} - \mathbf{v}_0|$ approach zero. The stream $\mathbf{v}_0(t, \mathbf{x})$ can be created by the motion of external bodies with surface S_0, as well as by singularities of the type of multipoles of arbitrary order.

The position and the deformation of bodies ∂V are defined by the generalized coordinates and the motion of the boundary of S_0 is given. The Lagrangian function L in (1.1) is then defined by formula

$$L = \int_{\Omega} \frac{\rho}{2} (\mathbf{v} - \mathbf{v}_0)^2 d\tau - \int_{V} p_0 \, d\tau \tag{2.2}$$

where $p_0(t, \mathbf{x})$ is the pressure in the stream $\mathbf{v}_0(t, \mathbf{x})$, and Ω, V are the regions occupied by fluid and the body respectively.

The result, when the stream $\mathbf{v}_0$ is induced by the motion of the surface, was obtained in /15/ by the indentical transformation of kinetic energy of the fluid appearing in the integral of action of variational equation of Thomson and Tait (1.1). For a small body the Lagrangian function is of the form /15/

$$L \approx T(\mathbf{x}^{\circ} - \mathbf{v}_0(t, \mathbf{x})) - p_0(t, \mathbf{x}) V \tag{2.3}$$

where $\mathbf{x}$ is the vector of coordinates of the geometric center of the body.

3°. Let in the complex plane z the velocity of the inhomogeneous stream and the stream function are determined by the formulas

$$\mathbf{v}_0 = \overline{dW_0/dz}, \quad \psi_0 = \operatorname{Im} W_0$$

where dW_0/dz is an arbitrary meromorphic function of the complex variable z whose coefficients are generally dependent on time t. The introduced into this stream simply connected contour $\partial\Omega$ who bounds from outside the region D free from singular points of function dW_0/dz.

The motion of contour $\partial\Omega$ is specified by the conformal mapping $z(t, \zeta)$ the exterior of unit circle $|\zeta| > 1$ on the exterior of contour $\partial\Omega$

$$z(t, \zeta) - z_0 = a_0(t)\zeta + a_2(t)/\zeta + a_3(t)/\zeta^2 + \ldots$$

The area of region D is assumed equal to constant quantity πa^2 which by the theorem of areas is expressed by condition $a_0^2 = a^2 + |a_2|^2 + 2|a_3|^2 + \ldots$

The motion of contour $\partial\Omega$ creates a perturbed potential velocity field $\mathbf{v} = \mathbf{v}_0 + \mathbf{V}_0 + \mathbf{V}_\Gamma$ which satisfies the conditions at the boundary of body D.

$$(\mathbf{V}_0 + \mathbf{v}_0)\mathbf{n}|_{\partial D} = v_n, \qquad \mathbf{V}_\Gamma \mathbf{n}|_{\partial D} = 0$$

Circulation $\mathbf{V}_0$ is zero, and $\mathbf{V}_\Gamma$ is equal Γ. At infinity both $\mathbf{V}_0$ and $\mathbf{V}_\Gamma$ approach zero.

The pressures p and p_0 satisfy Eqs.(1.4). Then the Lagrange function L in (1.1) is of the form /16/

$$L = \int_{\Omega} \frac{\rho}{2} V_0^2 d\tau - \int_{D} p_0 \, d\tau + \frac{\rho\Gamma^2}{8\pi} \ln \frac{a_0^2}{a^2} - \rho\Gamma \left(\frac{1}{2} \operatorname{Im}\left(z_0^{\cdot} \bar{z}_0 + \sum_{k=2}^{\infty} a_k^{\cdot} \bar{a}_k \right) + \frac{1}{2\pi i} \oint \psi_0 \frac{d\zeta}{\zeta} \right) \tag{2.4}$$

The variational equation (1.1) with function (2.4) conform to general formulas of Sedov for the force and moment acting on contour $\partial\Omega$ /13/

3. Case 1°. For the derivation of Lagrange function by formula (2.1) it is necessary first to define the velocity vector $\mathbf{v}_*$ /14/.

Let $q_1, q_2 \ldots$ be the generalized coordinates defining the position of boundary $\partial\Omega$ in space. The boundary $\partial\Omega$ consist of M connected parts $\partial\Omega_1, \partial\Omega_2 \ldots$ (Fig.1). Let some vector function $\mathbf{X}$

$$\mathbf{x} = \mathbf{X}(q_i, \mathbf{x}_*), \ \det \| \partial X_i/\partial x_j \| = 1$$

map region Ω_* onto region Ω ($\mathbf{x}_* \in \Omega_*, \mathbf{x} \in \Omega$) so that elementary volume is maintained at each point. The time functions $q_i(t)$ specify the motion of boundary $\partial\Omega$, while function $\mathbf{X}$ the trajectories of points $\mathbf{x} \in \Omega$, that correspond to $\mathbf{x}_* \in \Omega_*$.

It is thus possible to consider $\mathbf{x}_*$ as the Lagrangian coordinates of particles of some imaginary incompressible medium "fastened" with the boundary $\partial\Omega$. The velocity $\mathbf{v}_*$ and the Lagrangian displacement $\boldsymbol{\delta}_*\mathbf{x}$ of particles of that medium are determined by formulas

$$\mathbf{v}_* = \sum_{i=1}^{N} \frac{\partial \mathbf{X}}{\partial q_i} \dot{q_i}, \quad \boldsymbol{\delta}_*\mathbf{x} = \sum_{i=1}^{N} \frac{\partial \mathbf{X}}{\partial q_i} \delta q_i \tag{3.1}$$

On the boundary $\partial\Omega$ are satisfied the same conditions as for velocity $\mathbf{v}$ and the Lagrangian displacements $\boldsymbol{\delta}\mathbf{x}$ of particles of fluid

$$\mathbf{v}\mathbf{n} = \mathbf{v}_*\mathbf{n} = v_n, \quad \boldsymbol{\delta}\mathbf{x}\mathbf{n} = \boldsymbol{\delta}_*\mathbf{x}\mathbf{n} = \delta n$$

The simplest case is when the mapping function and velocity $\mathbf{v}_*$ are written in explicit form, it is an affine transformation of region Ω_* into Ω with determinant unity. Generally the affine transformation has 8 degrees of freedom, 6 for the solid body, and 4 for the mapping of plane region.

To obtain variational principle (1.1) it is necessary to express I in (1.6) in terms of functional variation. For the potential and vortex motion the respective results have the form

$$I(t_2) = \int_{t_1}^{t_2} dt \delta \sum_{k=1}^{M-1} \Gamma_k \gamma_k, \quad \gamma_k = \int_{\Pi_k} \rho(\mathbf{v} - \mathbf{v}_*)\mathbf{n}\, dS \tag{3.2}$$

$$I(t_2) = \int_{t_1}^{t_2} dt \delta \Big(\sum_{k=1}^{M-1} \Gamma_k \gamma_k + 2\rho\omega \int_{\Omega} (\psi - \psi_* - \psi_M + \psi_{M_*})\, d\tau \Big) \tag{3.3}$$

From this and (1.6) follow the formulas for the Laplace function

$$L = T - \Pi - \sum_{k=1}^{M-1} \Gamma_k \gamma_k - 2\rho\omega \int_{\Omega} (\psi - \psi_* - \psi_M + \psi_{M_*})\, d\tau \tag{3.4}$$

where 2ω is the intensity of vorticity $\operatorname{rot} \mathbf{v} = 2\omega\mathbf{k}$ for plane flow, $\operatorname{rot} \mathbf{v} = 2\omega y \mathbf{k}$ axisymmetric flow, $\mathbf{k}$ is the unit vector, y is the distance from the axis of symmetry, ψ and ψ_* are stream functions for the velocity fields $\mathbf{v}$, and $\mathbf{v}_*$, ψ_M, ψ_{M_*} are the values of stream functions $\partial\Omega_M$.

The equivalence of formulas (3.4) and (2.1) is proved using the transformations

$$\int_{\Omega} \rho(\mathbf{v} - \mathbf{v}_*)\mathbf{v}\, d\tau = \int_{\Omega} \operatorname{div} [\rho(\psi - \psi_*)\nabla\psi]\, d\tau - \int_{\Omega} \rho(\psi - \psi_*)\nabla^2\psi\, d\tau =$$

$$\sum_{k=1}^{M-1} \Gamma_k \gamma_k + 2\rho\omega \int_{\Omega} (\psi - \psi_* - \psi_M + \psi_{M_*})\, d\tau$$

$$L = \int_{\Omega} \rho \Big(\frac{v_*^2}{2} - \frac{(\mathbf{v} - \mathbf{v}_*)^2}{2} + U \Big) d\tau = \int_{\Omega} \rho \Big(\frac{v^2}{2} - (\mathbf{v} - \mathbf{v}_*)\mathbf{v} + U \Big) d\tau$$

The formula (2.1) for the Lagrange function has thus be obtained, Q.E.D.

4. Case 2°. The direct derivation of formula (2.2) for the Lagrange function may be obtained from the similar to (1.5) relation

$$\delta \Big[\operatorname{div} \Big(\frac{\rho}{2} \varphi \mathbf{V} \Big) \Big] = \frac{d}{dt} (\operatorname{div} \rho\varphi\boldsymbol{\delta}\mathbf{x}) + \operatorname{div} [(p - p_0)\, \boldsymbol{\delta}\mathbf{x}] \tag{4.1}$$

$$\mathbf{V} = \mathbf{v} - \mathbf{v}_0 = \operatorname{grad} \varphi$$

We integrated formula (4.1) over region Ω' bounded by the surface $\partial\Omega'$ consisting of the body boundary $\partial\Omega$, fairly withdrawn surface S_∞ and small spheres S_k surrounding singular points $\mathbf{x}_k$

$$\int_{t_1}^{t_2} dt\delta \int_{\partial\Omega} \frac{\rho}{2} \varphi \mathbf{V}\mathbf{n}\, dS = \int_{t_1}^{t_2} dt \Big(\int_{\partial\Omega} (p - p_0)\, \delta n\, dS + R \Big) + \int_{\partial\Omega} \rho\varphi\delta n\, dS \Big|_{t_1}^{t_2} \qquad (4.2)$$

By virtue of $\delta\mathbf{xn} = \delta\mathbf{n}$ and conditions (1.1) the last integral is zero.
The residual term R in (4.2) is expressed in terms of integral over surfaces S

$$R = -\delta \int_S \frac{\rho}{2} \varphi \mathbf{V}\mathbf{n}\, dS + \frac{d}{dt} \int_S \rho\varphi\delta\mathbf{xn}\, dS + \int_S (p - p_0)\, \delta\mathbf{xn}\, dS, \qquad S = S_\infty \cup \sum_k S_k \qquad (4.3)$$

For any closed surface S_k moving with the fluid, the expression (4.3) for R is identically zero. Then from (4.2) we obtain the variational equation (1.1) with Lagrange function (2.2).

To prove that $R \equiv 0$ we have to use formulas of differentiation and variation of the stream of vector over the closed surface moving with the fluid particles /10,17/

$$\frac{d}{dt} \int_S \mathbf{A}\mathbf{n}\, dS = \int_S \Big(\frac{\partial A}{\partial t} + \mathbf{v}\, \mathrm{div}\, \mathbf{A} + \mathrm{rot}\, (\mathbf{A} \times \mathbf{v}) \Big) \mathbf{n}\, dS \qquad (4.4)$$

$$\delta \int_S \mathbf{A}\mathbf{n}\, dS = \int_S (\partial \mathbf{A} + \delta\mathbf{x}\, \mathrm{div}\, \mathbf{A} + \mathrm{rot}\, (\mathbf{A} \times \delta\mathbf{x}))\, \mathbf{n}\, dS$$

where the symbols δ and ∂ denote, respectively, the Lagrangian and the Euler's variations, between which there exists the relation

$$\delta = \partial + (\delta\mathbf{x}, \nabla) \qquad (4.5)$$

Expressing the first two integrals in (4.3) in terms of formulas (4.4) and the last one in terms of Cauchy—Lagrange integral, we obtain

$$R = \int_S \rho \mathbf{f}\mathbf{n}\, dS, \qquad \mathbf{f} = -\frac{1}{2}\delta(\varphi\mathbf{V}) - \frac{1}{2}\delta\mathbf{x}\, \mathrm{div}\, (\varphi\mathbf{V}) + \qquad (4.6)$$

$$\varphi \frac{\partial\delta\mathbf{x}}{\partial t} - \frac{1}{2} \mathrm{rot}\, (\varphi\mathbf{V} \times \delta\mathbf{x}) + \mathbf{v}\, \mathrm{div}\, (\varphi\delta\mathbf{x}) + \mathrm{rot}\, (\varphi\delta\mathbf{x} \times \mathbf{v}) - \Big(\frac{v^2}{2} - \frac{v_0^2}{2} \Big) \delta\mathbf{x}$$

The partial derivative $\partial\delta\mathbf{x}/\partial t$ is found from the relation $d\delta\mathbf{x}/dt = \delta\mathbf{v}$ which with the help of (4.5) can be written in the form

$$\partial\delta\mathbf{x}/\partial t = \mathrm{rot}\, (\mathbf{v} \times \delta\mathbf{x}) + \partial\mathbf{V} \qquad (4.7)$$

If one takes into account that for incompressible fluid $\mathrm{div}\, \mathbf{V} = 0,\ \mathrm{div}\, \delta\mathbf{x} = 0$, and after some transformation, the integrand of (4.6) can be reduced to the form

$$\mathbf{f} = {}^1/_2 \varphi\delta\mathbf{V} - {}^1/_2 \mathbf{V}\partial\varphi - {}^1/_2\, \mathrm{rot}\, (\varphi\mathbf{V} \times \delta\mathbf{x})$$

The integral of the vector stream $\mathrm{rot}\, (\varphi\, \mathbf{V} \times \delta\mathbf{x})$ over the closed surface is identically zero, hence

$$R = \int_S \frac{\rho}{2} (\varphi\partial\mathbf{V} - \mathbf{V}\partial\varphi)\, \mathbf{n}\, dS$$

Functions $\varphi, \mathbf{V}, \partial\varphi, \partial\mathbf{V}$ are defined everywhere outside the body, the product $\varphi\partial\mathbf{V}, \mathbf{V}\partial\varphi$ decrease at infinity at least as r^{-3}, and besides $\mathrm{div}\, (\varphi\partial\mathbf{V} - \mathbf{V}\partial\varphi) \equiv 0$. From this it follows by the Gauss-Ostrogradskii theorem that $R \equiv 0$.

5. Case 3°. In the presence of circulation Γ the potential φ is nonhomogeneous. Hence it is necessary in integrating over the plane region and time t the equation (4.1) to apply the generalized theorem of Gauss — Ostrogradskii with allowance for the partition Π. In that case instead of Eq. (4.2) one can obtain

$$\int_{t_1}^{t_2} dt\delta \int_{\Omega'} \frac{\rho}{2} V^2 d\tau = \int_{t_1}^{t_2} dt \Big(\int_{\partial\Omega} (p - p_0)\, \delta n\, dS + G \Big) + \rho\Gamma \int_\Pi \delta\mathbf{xn}\, dl \qquad (5.1)$$

The residual term G is found similarly to (4.4)

$$G = \frac{d}{dt} \int_c \rho\varphi\delta\mathbf{xn}\, dl + \int_c (p - p_0)\, \delta\mathbf{xn}\, dl, \quad c = c_\infty + \sum_k c_k \qquad (5.2)$$

Vector $\delta\mathbf{x}$ by virtue of the equation of incompressibility can be expressed in terms of scalar function g using formula $\delta\mathbf{x} = \operatorname{rot} \mathbf{k}g$, where $\mathbf{k}$ is a unit vector normal to the plane. Equation (4.7) has the integral

$$dg/dt = \partial\psi \tag{5.3}$$

where ψ is the stream function ($\mathbf{v} = \operatorname{rot} \mathbf{k}\psi$). It follows from (5.3) that g is a bounded function.

If one takes into account the Cauchy–Lagrange integral, formulas (5.3) and the relation $\delta\mathbf{x}\mathbf{n}dl = dg$, then from (5.2) we can obtain that

$$G = \int_c \rho\left(\frac{V^2}{2} dg - \partial\psi\, d\varphi\right) \tag{5.4}$$

Using the properties of decrease of functions $\mathbf{V}$, $\partial\psi$, φ at infinity, it is possible to show that when contour c_∞ recedes at infinity and, contours c_k contract to singular points z_k, for the integrals (5.1) and (5.4) are obtained the following limit expressions /16/:

$$\delta \int_{\Omega'} \frac{\rho}{2} V^2 d\tau \to \delta\left(\int_{\Omega} \frac{\rho}{2} V_0^2 d\tau - \frac{\Gamma^2}{8\pi} \ln \frac{a_0^2}{a^2}\right)$$

$$\int_{\Pi} \rho\Gamma\delta\mathbf{x}\mathbf{n}\, dl \to \int_{t_1}^{t_2} dt\delta\left(\rho\Gamma \operatorname{Im} \frac{1}{2} \sum_{n=1}^{\infty} a_n^{\cdot} \bar{a}_n - \frac{\rho\Gamma^2}{4\pi} \ln \frac{a_0^2}{a^2} + \frac{\rho\Gamma}{2\pi} \oint \psi_0 \frac{d\zeta}{i\zeta}\right), \quad G \to 0$$

Passing to the respective limit in (5.1) we obtain for the Lagrange function Eqs.(1.1) the final form (2.4).

The Lagrange function in a stream with constant vorticity can be similarly obtained from the following equation for variations /17/:

$$\delta \frac{\rho}{2} V^2 = \rho \frac{d}{dt} (\mathbf{V}\delta\mathbf{x}) + \operatorname{div} [(p - p_0)\delta\mathbf{x}] + \rho(\operatorname{rot} \mathbf{v}_0, \mathbf{V}, \delta\mathbf{x})$$

6. Examples of Dynamical systems. When the Lagrange function $L(q_i, q_i^{\cdot})$ is computed in the explicit form, the problem of hydrodynamic reduces to the conventional system of Lagrangian dynamics, which enables us to produce the complete mathematical analysis of non-linear equations using methods of analytic mechanics. This method was approved in investigations of the general problem of motion of solid body with fluid /10–12/, and has shown a high effectiveness, particularly in the analysis of stability of steady motions with allowance for capillary effects.

Formulas (2.1)–(2.4) widen the class of hydrodynamic systems that reduce to Lagrangian dynamics.

The problem of motion of body in inhomogeneous stream and the hydrodynamic interaction of bodies was investigated on the basis of (2.2), and (2.3) in /15/. General equations of motion were obtained for a solid as well as for deformable bodies (e.g., the bubble /18/) in a inhomogeneous stream. And the equations are no more complex than Kirchhoff's /2/ and are well suited for practical computations. The instability of steady motion of solid in an inhomogeneous stream is proved in /19/. The prospect is open for investigation by the direct Liapunov method of the problem of stability of steady motion of a bubble or drop /14,20/.

The combination of formulas (2.1) and (2.4) enables us to write the ordinary differential equations that determine the dynamics of vortex distribution in a given stream. This approach was used for constructing model of the Golfstream ring /21/.

The variational formulation of the problems of hydrodynamics provides the possibility to apply straight method of determining the free boundary or the boundary separating the vortex and potential flows /14,20,21/.

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